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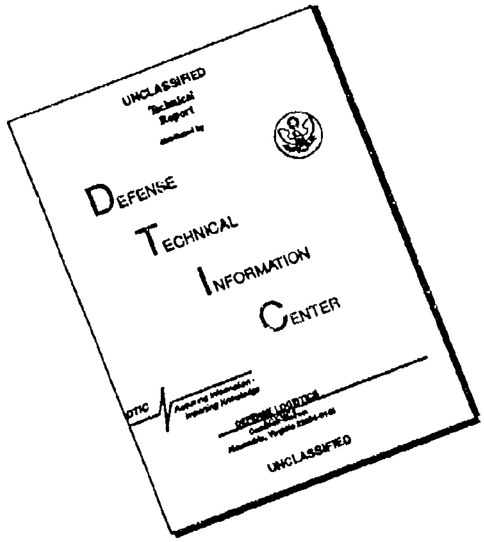

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A MULTIPLE-ASSIGNMENT PROBLEM

by

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SUMMARY

A generalization of Kuhn's simple assignment problem is considered: There are m men and n tasks given with each man qualified for certain of the tasks. The output from each task is given as a concave function of the number of qualified men assigned to it. Find an assignment of men to tasks, perhaps more than one man to a task, so as to maximize total output.

An algorithm for solving this general problem is given in which transfers like those used by Kuhn on the simple problem are selected using a node-labeling procedure on a related network. The algorithm yields for every k, $1 \leq k \leq m$, an optimal assignment of the first k men only, employing a single transfer to increase k by one. Several special forms of the generalized problem are considered including a target-assignment problem which A. S. Manne has formulated as a linear program.

§1. INTRODUCTION

In solving the general (linear) assignment problem, Kuhn considers a simple assignment problem [2; p. 83], in which it is required to assign m men to n tasks, one to each task, so that a maximum number of men are assigned to tasks for which they qualify. In treating the simple problem, he considers partial assignments in which each man is either unassigned or assigned to a task for which he qualifies and shows that a sufficient number of transfers transform any partial assignment into an optimal one. A transfer consists of a "bumping" operation in which an unassigned man is assigned to a task for which he qualifies, releasing a second man to take a different task for which he qualifies, releasing a third, etc., ending with a k^{th} man who is moved to an unoccupied task.

In §2 we introduce a generalization of the simple assignment problem, a multiple-assignment problem, in which m men are to be assigned to n tasks, perhaps more than one per task. We assume the output from a task is a concave function of the number of qualified men assigned to it and ask for an assignment which maximizes total output. In §3 we describe an algorithm for the multiple-assignment problem utilizing transfers and show it yields an optimal assignment. Finally, in §4 we consider three special cases of the multiple-assignment problem: the simple assignment problem itself, an assignment problem with priorities on the tasks, and a target-assignment problem described by Manne [3].

§2. DEFINITIONS AND PRELIMINARIES

A formal statement of the multiple-assignment problem is contained in the following three definitions:

THE QUALIFICATION MATRIX, Q. An $n \times m$ matrix $Q = \{q_{ij}\}$ is a <u>qualification matrix</u> if it is a (0,1)-matrix (a matrix with entries 0 or 1 only) without zero columns.

THE ASSIGNMENT MATRIX, X. An $n \times m$ matrix $X = \{x_{ij}\}$ is an <u>assignment</u> for the $n \times m$ qualification matrix Q if it is a (0,1)-matrix with exactly one nonzero entry per column and

(1) $\quad x_{ij} = 0 \quad$ whenever $\quad q_{ij} = 0.$

THE MULTIPLE-ASSIGNMENT PROBLEM, $[Q,f_i]$. An $n \times m$ qualification matrix Q and a set of n concave functions $f_i(k)$, $1 \leq i \leq n$, defined for integral k, are given. An assignment X for Q is to be found which maximizes

(2) $\quad \varphi(X) = \Sigma_i f_i(\Sigma_j x_{ij}).$

In terms of men and tasks, $q_{ij} = 1$ if and only if man j qualifies for task i, $x_{ij} = 1$ if and only if man j is assigned to task i, $\Sigma_j x_{ij}$ is the number of men assigned to task i, and $f_i(k)$ is the output from task i if k (qualified) men are assigned to it.

The formulation stated assumes each man is qualified for at least one task, each man is assigned to some task, and no man is assigned to a task for which he does not qualify. These are not material restrictions. If an application should arise in which a man need not be assigned to a task for which he qualifies, an additional, <u>idle</u> task may be introduced for which all men qualify but for which the output is zero. Also there is no essential restriction in supposing the values $f_i(k)$ are defined for all integers k, though clearly only the values $f_i(k)$ for k from

0 to m can have any effect on the solution.

For convenience in discussing the problem, we define

(3) $$r_i(Z) = \Sigma_j z_{ij}$$

for any matrix $Z = \{z_{ij}\}$, and

$$\delta_i(k) = f_i(k+1) - f_i(k), \qquad 1 \leq i \leq n.$$

Also, if X is an assignment for a problem $[Q,f_i]$ we define

$$\delta_i^+(X) = \delta_i(r_i(X))$$

and

$$\delta_i^-(X) = \delta_i(r_i(X) - 1)$$

which are respectively the gain and loss from adding a man to or (provided $r_i(X) \neq 0$) removing a man from task i. Note that the concavity of the f_i is equivalent to the condition

(4) $$\delta_i(k) \geq \delta_i(k^*) \quad \text{if} \quad k \leq k^*.$$

Given any two assignments X and Y for a problem $[Q,f_i]$, their difference, $D = Y - X$, may be consid ·ed the incidence matrix of a directed, loopless graph, G, with a node for each row of D and an arc for each nonzero column of D. The arc corresponding to the nonzero column j of D starts at node i^* and ends at node i, where

$$d_{i^*j} = -1, \qquad d_{ij} = 1.$$

A path of length k from node a to node b of G is a sequence of k arcs visiting in succession $k+1$ distinct nodes starting with a and ending with b. A circuit is a path except that the starting and ending nodes only are identical. We will say a collection of matrices of dimension $n \times m$ are <u>disjoint</u> if for each j, $1 \leq j \leq m$, at most one matrix has a nonzero entry in column j. It will be convenient to refer to the incidence matrices of graphs using graph terminology. Thus, P below is a path of length 4 from row 1 to row 4, C is a circuit of length 3, and P and C are disjoint.

$$P = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

The proofs of the following lemmas are not difficult and are omitted.

LEMMA 1. If X and Y are assignments for $[Q,f_i]$ and $P = Y - X$ is a path from a to b, then

$$\varphi(Y) - \varphi(X) = \delta_b^+(X) - \delta_a^-(X).$$

LEMMA 2. If X and Y are assignments for Q, the difference $D = Y - X$ can be written as a finite sum

$$D = P_1 + P_2 + \cdots + C_1 + C_2 + \cdots, \tag{5}$$

where the P_i and C_i are disjoint matrices, the C_i are circuits, and the P_i are paths with a maximal property, namely:

(6) No path ends at a node (row) where another path starts.

Moreover, if E is an arbitrary sum of paths and circuits in (5), then X+E is an assignment for Q.

§3. THE ALGORITHM

In order to simplify the statement of the algorithm for solving the multiple-assignment problem, we introduce the notion of a special task. Task s in a multiple-assignment problem $[Q,f_i]$ is a <u>special task</u> if for some γ

$$(7) \qquad f_s(k) = k \cdot \gamma ,$$

i.e. if f_s is linear, and all men qualify for task s. In particular, an idle task is a special task.

Given any problem $[Q,f_i]$ which does not already contain a special task we may introduce one by adding a row of 1's to Q and defining the corresponding output function by (7). It is clear that if γ is chosen sufficiently negative, in particular if

$$(8) \qquad \gamma < \delta_i(k), \qquad\qquad i \neq s, \quad 0 \leq k \leq m,$$

then an optimal assignment for the augmented problem will assign no men to the special task and therefore will (on deletion of row s) be an optimal assignment for the original problem.

The following lemma provides the foundation for the algorithm.

LEMMA 4. Suppose Q is an $n \times m$ qualification matrix and $[Q,f_i]$ is a multiple-assignment problem with special task s. Let Q^ℓ, $0 \leq \ell \leq m$, be the qualification matrix equal to Q in the first ℓ columns and in row s, but zero elsewhere. Suppose for some k, $1 \leq k \leq m$, that X^{k-1} is an optimal assignment for $[Q^{k-1},f_i]$. Let X^* be chosen from the optimal assignments for $[Q^k,f_i]$ so as to minimize the number of arcs (nonzero columns) of $D = X^* - X^{k-1}$.

Then $D = 0$ or D is a path starting at s.

Proof. If $D = 0$, there is nothing to prove. If D is not the zero matrix, it may be decomposed as in Lemma 2. Since

$$r_i(\Sigma_j C_j) = 0,$$

it follows that

$$r_i(\Sigma_j P_j) = r_i(D)$$

or

$$\varphi(X^{k-1} + \Sigma_j P_j) = \varphi(X^*).$$

But $\Sigma_j P_j$ would consist of fewer arcs than D if the decomposition, (5), involved any circuits. Hence, by the choice of X^*, $D = \Sigma_j P_j$.

Suppose (5) includes a path P, from a to b, which does not involve column k. Let $E = D - P$. We do not exclude the possibility $E = 0$. By Lemma 2, $X^{k-1} + E$ is an assignment for $[Q^k, f_i]$, but $X^* = X^{k-1} + E + P$ is optimal, so

$$\varphi(X^{k-1} + E + P) - \varphi(X^{k-1} + E) \geq 0. \tag{9}$$

In fact, by the choice of X^*, equality cannot hold in (9). Similarly

$$\varphi(X^{k-1} + P) - \varphi(X^{k-1}) \leq 0, \tag{10}$$

since $X^{k-1} + P$ is an assignment for $[Q^{k-1}, f_i]$ as well as for $[Q^k, f_i]$. By Lemma 1, (9) (with inequality) and (10) imply

$$\begin{aligned} &\delta_b^+(X^{k-1} + E) - \delta_a^-(X^{k-1} + E) > 0 \\ &\delta_b^+(X^{k-1}) - \delta_a^-(X^{k-1}) \leq 0 . \end{aligned} \tag{11}$$

Since by (6) no path of E ends at a or begins at b, it follows

$$(12)\qquad \begin{aligned} r_b(X^{k-1} + E) &\geq r_b(X^{k-1}) \\ r_a(X^{k-1} + E) &\leq r_a(X^{k-1}), \end{aligned}$$

which by concavity, as given in (4), implies

$$(13)\qquad \begin{aligned} \delta_b^+(X^{k-1} + E) &\leq \delta_b^+(X^{k-1}) \\ \delta_a^-(X^{k-1} + E) &\geq \delta_a^-(X^{k-1}). \end{aligned}$$

But equations (11) and (13) are incompatible. Hence every path in (5) must involve column k, and as (5) is a disjoint sum, D is itself a path involving column k.

Suppose now that D does not start at the special task, i.e. $a \neq s$. Since D involves column k, it visits s but does not end there. D, then, may be written as a sum of two disjoint paths: R from a to s, which does not involve column k, and T from s to b. We would like to repeat the arguments of the preceding paragraph with $D = R + T$ in place of (5) and R in place of P. Unfortunately, R and T do not satisfy (6). However, (12) is violated only for the special task, for which (13) holds anyway. The rest of the argument applies, leading to a contradiction. This proves the lemma.

From the foregoing and Lemma 1, the following theorem is easily proven. As mentioned before, there is no restriction in supposing $[Q,f_i]$ involves a special task.

THEOREM 1. Suppose $[Q,f_i]$ is a multiple-assignment problem with special task s. Let Q^ℓ, $0 \leq \ell \leq m$, be as defined in Lemma 4. Suppose X^0 is Q^0 and for each k, $1 \leq k \leq m$, X^k is chosen from among the assignments for Q^k which differ from X^{k-1} by a path from row s to row b (X^{k-1} is considered such an assignment with $b = s$) so as to maximize $\delta_b^+(X^{k-1})$. Then each X^k is an optimal assignment for $[Q^k, f_i]$ and X^m is an optimal assignment for $[Q, f_i]$.

In order to apply Theorem 1 in practice, a method must be available for choosing row b for each k and constructing X^k. For each k, Q^k and X^{k-1} may be thought of as defining a directed graph with rows of Q^k as vertices and an arc from row i^* to row i for each triple, (i, i^*, j), for which

(14) $$x_{i^*j}^{k-1} = q_{i^*j}^{k-1} = 1, \qquad q_{ij}^{k-1} = 1, \qquad i \neq i^*.$$

Each arc indicates the possibility of shifting man j, $j \leq k$, from task i^* to task i. Each path in the graph so constructed corresponds exactly to a path P (in the matrix sense) for which $X^{k-1} + P$ is an assignment for Q^k. If a path starts at s it corresponds to what Kuhn terms a transfer.

A familiar procedure for constructing paths in networks (see, for example, Ford and Fulkerson [1, pp. 17-18]) can be adapted for choosing b and the path $X^k - X^{k-1}$ as follows: Initially, all rows are unlabeled except row s, which has label $(0,0)$. For any (i, i^*, j) satisfying (14) such that row i^* is labeled but row i is not, the label (i^*, j) is attached to row i. When no more rows can be labeled, a row, b, is chosen from the labeled rows so as to maximize

$\delta_b^+(X^{k-1})$. If $b = s$, X^{k-1} is taken for X^k. If $b \neq s$, a finite sequence of labels,

$$(15) \qquad (i_1,j_1),\ (i_2,j_2),\ \ldots,\ (i_t,j_t),\ (0,0)$$

is generated starting with the label of row $i_0 = b$ according to the rule (i_h,j_h) is the label of row i_{h-1}. By the nature of the labeling process, all i_h are distinct, as are all j_h. In this case, X^k is formed from X^{k-1} by moving the 1 in column j_h of X^{k-1} from row i_h to row i_{h-1} for each h, $1 \leq h \leq t$. In practice X^k can be formed at the time (15) is generated.

Regardless of the sequence of labeling, the same set of rows are eventually labeled. Also, any optimal assignment X^k for $[Q^k,f_i]$ which differs from X^{k-1} by a path from s can be obtained by suitable labeling order and choice of b maximizing $\delta_b^+(X^{k-1})$.

In practice, a row may be given the label j instead of (i^*,j) since i^* can be determined from X^{k-1}. Also, the backtrack, (15), may be started as soon as a row, b, is labeled for which in some way it is known

$$\delta_b^+(X^{k-1}) = \beta = \max_{1 \leq a \leq n} \delta_a^+(X^{k-1}) .$$

§4. THREE EXAMPLES

In this section, we consider special cases and applications of the multiple-assignment problem.

We note first that the simple assignment problem is a special case of the multiple problem. The qualifications of the m men in a simple problem determine an $m \times m$ square qualification matrix Q to which may be added an $m+1^{st}$ row consisting of 1's to obtain an $(m+1) \times m$ qualification matrix Q^*. If we define

$$f_i(0) = 0, \qquad 1 \le i \le m$$

$$f_i(k) = 1, \qquad 1 \le i \le m, \qquad k \ge 1,$$

$$f_{m+1}(k) = k \cdot \gamma,$$

then $[Q^*,f_i]$ is a multiple-assignment problem with special task $m + 1$. Application of the algorithm of Theorem 1 to $[Q^*,f_i]$ for different values of γ will yield different optimal assignments, $X(\gamma)$. However, since the transfers (paths, $X^k - X^{k-1}$) employed in executing the algorithm if $0 < \gamma < 1$ would be suitable in each case as transfers if $\gamma = 0$, it follows that an assignment $X(\frac{1}{2})$, say, is also an optimal assignment with $\gamma = 0$. Since $X(\frac{1}{2})$ must assign at most one man to any task except the $m+1^{st}$, it is clear it provides an optimal partial assignment for the simple assignment when men in the $m+1^{st}$ task are considered unassigned. If labeling is discontinued as soon as a row, b, with $\delta_b^+(X^{k-1}) = 1$ is labeled, the multiple-assignment algorithm is essentially the one given by Kuhn.

Consider next a multiple-assignment problem $[Q,f_i]$ (without idle task) such that

$$(16) \qquad \begin{aligned} f_i(k) &= c_i k, & k &\le m_i \\ f_i(k) &= c_i m_i & k &\ge m_i \end{aligned}$$

and

$$(17) \qquad c_1 \gg c_2 \gg \cdots \gg c_n > 0.$$

Provided the inequalities (17) are sufficiently great, this problem is equivalent to finding a best assignment of men to prioritized tasks where task i has a quota of m_i men. The usual difficulty in achieving a solution to a priority-type problem using a system of inequalities such as (17) is the extreme range required in the size of the c_i. Here, however, we find:

THEOREM 2. If X is an optimal assignment for $[Q, f_i]$ where the f_i are given by (16), and

(18) $$c_1 > c_2 > \cdots > c_n > 0,$$

then X is also optimal for any choice of the c_i satisfying

(19) $$c_1 \geq c_2 \geq \cdots \geq c_n \geq 0.$$

Proof. This result is immediate on noting that the values $\delta_i(X^{k-1})$ which determine permissible transfers in the algorithm of Theorem 1 are exactly the quantities in (18) and hence the same transfers are permissible if the c_i's satisfy (19).

This theorem has two consequences of practical significance. First, it is sufficient for solving the priority problem to use, say, the first n integers for the c_i, and second, nothing is sacrificed by putting priorities on the tasks -- an optimal assignment with priorities is also an optimal assignment without them, i.e. with $c_i = 1$, $1 \leq i \leq n$.

Finally, we will show that a target-assignment problem considered by Manne [3] can be handled efficiently as a multiple-assignment problem. Suppose m guns are available to direct against n targets. Some guns cannot be directed to certain targets, but all guns which can be directed against a specific target are equally effective. Each target i, $1 \leq i \leq n$, has a value a_i and the expected value of this target after a specified time of bombardment is

$$(20) \qquad \bar{a}_i = a_i \prod_{j=1}^{m} (1 - p_i x_{ij}),$$

where p_i is the probability that one gun alone will destroy the target and x_{ij} is the probability (after the manner of game-theoretic strategies) that gun j is to be directed at target i. It is desired to choose the x_{ij}, subject to

$$0 \leq x_{ij} \leq 1, \qquad 1 \leq i \leq n, \quad 1 \leq j \leq m,$$

$$\Sigma_i x_{ij} = 1, \qquad 1 \leq j \leq m,$$

and the restriction that $x_{ij} = 0$ if gun j cannot be directed against target i, so as to minimize $A = \Sigma_i \bar{a}_i$. A little thought shows that for each j, A is linear in the variables $x_{1j}, x_{2j}, \ldots, x_{nj}$ and hence there is no essential restriction in supposing the x_{ij} are integral.[1] If we look only at integral assignments, (20) may be replaced by

$$(21) \qquad a_i \Pi_j (1 - p_i)^{x_{ij}} = a_i (1 - p_i)^{\Sigma_j x_{ij}} .$$

It is clear now that the problem is a multiple-assignment problem if we set

$$f_i(k) = a_i [1 - (1 - p_i)^k]$$

since $f_i(k)$, the expected loss in value of target i under fire from k guns, is concave in k.

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[1] This is a point which Manne apparently misses in his discussion of the problem. There is a question whether (21) ought properly to be used in place of (20) with x_{ij} being the portion of time gun j is directed at target i. However, it is not our aim to compare models here.

Manne shows that the problem (with assumption of integral assignments) is a linear program of the transportation type. It might be asked if the speed of algorithms for solving transportation linear programs makes Manne's formulation competitive with the algorithm of Section 3. The answer seems to be 'no' for the following reason: The linear program is highly degenerate so that many pivots are required before one is found which decreases the objective. The pivots which do decrease the objective, however, correspond roughly to improving transfers in the multiple-assignment algorithm.

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